

## HAMILTONIAN PRISMS ON 5-CHORDAL GRAPHS

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ABSTRACT. In this paper, we provide a method to find a Hamiltonian cycle in the prism of a 5-chordal graph, which is  $(1 + \epsilon)$ -tough, with some special conditions.

## 1. PRELIMINARIES

The toughness of graphs is a concept introduced by Chvátal [6], when he was doing research on Hamiltonicity of graphs. A graph  $G$  is called  $\beta$ -tough, if for any  $p \geq 2$ , it cannot be split into  $p$  components by deleting less than  $n\beta$  vertices. It is not difficult to prove that every  $k$ -tough graph is  $2k$ -connected. A graph  $G$  is called  $k$ -chordal, if for any circle  $C$  in  $G$  with length  $|C| \geq k$ ,  $C$  has a chord in  $G$ . Usually, we call a 3-chordal graph a chordal graph for convenience. Clearly, a chordal graph is also  $k$ -chordal for any  $k \geq 3$ .

Chvátal posed a famous conjecture, which is still open today, saying that there exists a constant  $\beta$  such that every  $\beta$ -tough graph is Hamiltonian. Clearly, being 1-tough is a necessary condition for being Hamiltonian. What is more, there exist 2-tough graphs which are not Hamiltonian [1]. In recent decades, researchers found many special classes of graphs for which Chvátal's conjecture is true, for example, chordal graphs [9],  $2K_2$ -free graphs [5], and planar graphs [12]. Moreover, researchers are also interested in many kinds of analogies of Hamiltonian cycles, such as  $k$ -walks, Hamiltonian-prisms, and 2-factors. A  $k$ -walk in a graph  $G$  is a closed walk visiting each vertex of  $G$  at least once but at most  $k$  times. Clearly, a Hamiltonian cycle can be considered as a 1-walk. And a  $p$ -walk is trivially a  $q$ -walk, for integers  $p \leq q$ . The *prism* over a graph  $G$  is the Cartesian product  $G \times K_2$  of  $G$  with the complete graph  $K_2$ . If  $G \times K_2$  is Hamiltonian, then we say  $G$  is *prism-Hamiltonian*, and we call  $G \times K_2$  the *Hamilton-prism* of  $G$ . It is not difficult to prove that being prism-Hamiltonian is a property stronger than admitting 2-walk but weaker than being Hamiltonian [10]. For  $k$ -walks in graphs, there is also a well-known open conjecture, which is posed by Jackson and Wormald, saying that every  $\frac{1}{k-1}$ -tough graph admits a  $k$ -walk [8]. A *2-factor* in a graph is a spanning subgraph, which is consisted of several disjoint cycles. An *edge-dominating cycle*  $C$  in a graph  $G$  is a cycle such that the induced subgraph on  $V(G) - V(C)$  contains no edge.

Here we list several well-known results relative chordal graphs and 5-chordals on the topic of Hamiltonicity.

**Theorem 1.** [9] Every 10-tough chordal graph is Hamiltonian.

**Theorem 2.** [1] There exists a  $(\frac{7}{4} - \epsilon)$ -tough chordal non-Hamiltonian graphs, for any  $\epsilon > 0$ .

**Theorem 3.** [3] Every  $(1 + \epsilon)$ -tough chordal planar graph is Hamiltonian, for any  $\epsilon > 0$ .

**Theorem 4.** [11] Every  $(\frac{3}{4} + \epsilon)$ -tough chordal planar graph admits a 2-walk, for any  $\epsilon > 0$ .

**Theorem 5.** [2] Every  $\frac{3}{2}$ -tough 5-chordal graph contains a 2-factor.

## 2. MAIN RESULTS

The main result in this paper is following:

**Theorem 6.** Let  $G$  be a  $(1 + \epsilon)$ -tough 5-chordal graph, if  $G$  contains an edge-dominating cycle, then  $G$  is prism-Hamiltonian.

The proof of this theorem is consisted of the following two lemmas, one of which is from [7], while another is new.

**Lemma 1.** [7] Let  $G$  be  $(1 + \epsilon)$ -tough, for some  $\epsilon > 0$ .

- (1) If  $G$  contains an edge-dominating cycle  $C$  with even number of vertices, then the prism over  $G$  is Hamiltonian.
- (2) If  $G$  contains an edge-dominating cycle  $C = v_1v_2 \cdots v_{2p+1}v_1$  of odd length, and there are three vertices  $v_1, v_{2q}$  and  $v_{2q+1}$ , for some  $1 \leq q \leq p$ , inducing a triangle in  $G$ , then the prism over  $G$  is Hamiltonian.

**Lemma 2.** Assuming  $C = v_1v_2 \cdots v_pv_1$  is a cycle of odd length in a 5-chordal  $G$ , then there exist three vertices  $v_i, v_{i+2q-1}, v_{i+2q}$  of  $C$  (the index is in module  $p$ ), which induce a triangle in  $G$ .

*Proof.* If  $p = 3$ , then let  $i = 1$  and  $q = 1$ , so the result holds trivially. If  $p \geq 5$ , then by 5-chordality, there must be a chord on  $C$ . Without loss of generality, we can assume that  $v_1$  is one endpoint of a chord  $e$ . If another endpoint of  $e$  is  $v_3$ , then we have proved the result. Otherwise,  $e = v_1v_t$  divides  $C$  into two cycles:  $C_1 = v_1v_2 \cdots v_tv_1$  and  $C_2 = v_1v_t \cdots v_{t+1}v_pv_1$ .

If  $t$  is odd, then by mathematical induction, we can find triangle with the property we want in  $C_1$ .

If  $t$  is even, then we can relabel the vertices of  $C_2$  by  $v'_1 = v_1, v'_2 = v_t, v'_3 = v_{t+1}$  and so on. Clearly, this relabelling process does not change the odd-even of the index. So, by mathematical induction, there is a triangle we want in  $C_2$ , and this triangle is also the one we want in  $C$ . □

Combining the two lemmas above, we have finished the proof of Theorem 6

## 3. APPLICATIONS

Now, we are interested in the question that under what condition, we can find an edge-dominating cycle in a 5-chordal graph? Here we list several well-known results on this topic.

**Lemma 3.** [4] Let  $G$  be a 2-connected graph of order  $n$ . If

$$\delta_3(G) := \min\left\{\sum_{i \leq 3} d(x_i) \mid x_1, x_2, x_3 \text{ are independent vertices in } G\right\} \geq n + 2,$$

then all longest cycles in  $G$  are edge-dominating.

Two edges are called *remote* if they are disjoint and there is no edge joining them. Let  $N_G(v)$  stand for the set of vertices in  $G$  which are adjacent to  $v$ . For an edge  $e = uv$  in  $G$ , we define the *degree*  $d(e)$  of  $e$  by  $d(e) = |N_G(u) \cup N_G(v) - \{u, v\}|$ .

**Lemma 4.** [13] Let  $G$  be a  $k$ -connected graph ( $k \geq 2$ ) such that, for every  $k + 1$  mutually remote edges  $e_0, e_1, \dots, e_k$  of  $G$ ,

$$\sum_{i=0}^k d(e_i) > \frac{1}{2}k(|V(G)| - k).$$

Then  $G$  contains an edge-dominating cycle.

**Lemma 5.** [14] Let  $G$  be a 2-connected graph. If  $d(e_1) + d(e_2) > |V(G)| - 4$  for any remote edges  $e_1, e_2$ , then all longest cycles in  $G$  are edge-dominating cycles.

Combining the three lemmas above and Theorem 6, we get the following corollaries.

**Corollary 1.** Let  $G$  be a  $(1 + \epsilon)$ -tough 5-chordal graph of order  $n$ . If

$$\delta_3(G) \geq n + 2,$$

then  $G$  is prism-Hamiltonian.

**Corollary 2.** Let  $G$  be a  $(1 + \epsilon)$ -tough 5-chordal graph such that, for every 3 mutual remote edges  $e_1, e_2, e_3$  of  $G$ ,

$$\sum_{i=0}^3 d(e_i) > \frac{3}{2}(|V(G)| - 3).$$

Then  $G$  is prism-Hamiltonian.

**Corollary 3.** Let  $G$  be a  $(1 + \epsilon)$ -tough 5-chordal graph. If  $d(e_1) + d(e_2) > |V(G)| - 4$  for any remote edges  $e_1, e_2$ , then  $G$  is prism-Hamiltonian.

**Remarks 1.** The results in this paper is almost trivial now. So, we are looking for some “non-trivial” condition for 5-chordal graphs to have edge-dominating cycles. Especially, we hope to find a toughness condition for 5-chordal graphs to have such cycles.

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